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Nonlinear Bénard convection with rotation

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The equations for nonlinear Bénard convection with rotation for a layer of fluid, thickness d, are derived using the Glansdorff & Prigogine (1964) evolutionary criterion as used by Roberts (1966) in his paper on non-rotational Bénard convection. The parameters of the problem in this case are the Rayleigh number $R = \alpha g \Delta\theta d/\nu K$, the Taylor number $T = 4d^4\Omega_3^2/\nu^2$ and the Prandtl number $Pr = \nu/K$, where α is the coefficient of volume expansion, g the acceleration due to gravity, $\Delta\theta$ the temperature difference between the horizontal surfaces, ν the kinematic viscosity, K the thermal diffusivity and Ω_3 the rotation rate about the vertical direction. The asymptotic solution for two-dimensional cells (rolls) is investigated for large Rayleigh numbers and large Taylor numbers. For rolls the convection equations are found to be independent of the Prandtl number. However, the solutions depend upon the Prandtl number for another reason. The rotational problem differs from the non-rotational one in that the Rayleigh number and the horizontal wavenumber a of the convection are now functions of the Taylor number. These are taken to be $R \sim \rho T^{\alpha'}$ and $a \sim AT^{\beta}$, where α' and β are positive numbers. Thermal layers develop as R becomes large with ρ or T becoming large. The order in which ρ and T are allowed to increase is important since the horizontal wavenumber a also increases with T and the convection equations can be reduced in this case. A liquid of large Prandtl number such as water has $\nu \gg K$. Since $R \sim O(1/\nu K)$ and $T \sim O(1/\nu^2)$, ρ will be greater than T for a given (large) $\Delta\theta$ and Ω_3. Similarly, for a liquid of small Prandtl number such as mercury $\nu \ll K$, and T is greater than ρ for a given $\Delta\theta$ and Ω_3. For rigid–rigid horizontal boundaries with ρ large and then T large the ρ thermal layer has the same structure as for the non-rotating problem. As $T \to \infty$ three types of thermal layers are possible: a linear Ekman layer, a nonlinear Ekman layer and a Blasius-type thermal layer. When the horizontal boundaries are both free the ρ thermal layer is again of the same structure as for non-rotating Bénard convection. As $T \to \infty$ a nonlinear Ekman layer and a Blasius-type thermal layer are possible.

When T is large and then ρ made large the differential equations governing the convection are reduced from eighth order to sixth order owing to a becoming large as $T \to \infty$. There are Ekman layers as $T \to \infty$, when the horizontal boundaries are both rigid. The ρ thermal layers now have a different structure from the non-rotating problem for both rigid–rigid and free–free horizontal boundaries. The equation for small amplitude convection near to the marginal case is derived and the solution for free–free horizontal boundaries is obtained.

1. Introduction

In this paper the equations for steady convection are derived using the Glansdorff–Prigogine (1964) evolutionary criterion. Nonlinear overstability and the preferred-mode problem are not discussed. The technique of Glansdorff & Prigogine is equivalent to a particular form of Galerkin's method, but as pointed out by Roberts (1966), their evolution criterion contains information which is beyond the scope of Galerkin's method. However, it may be difficult to profit from this information in practice.

In particular, the asymptotic theory for large Rayleigh numbers R is investigated for two-dimensional cells (rolls). According to Rossby (1969) these are the preferred cell shapes for steady convection. As in the non-rotational case the temperature θ is expanded as $\theta = \theta_0(z) + F(z)f(x,y)$. For rolls the approximation $f(x,y) = 2\cos ax$ is used, where a is the horizontal wavenumber. Notice that the full expansions $\theta_0(z)$ and $F(z)$ are used for the vertical direction. A criticism of this is that it is both mathematically inconsistent to keep the infinite number of z terms and only two x terms and physically unrealistic to attempt thereby to keep the boundary-layer structure on the horizontal walls while ignoring the narrowness of the vertical 'plumes'. This point was made by Tritton & Zarroga (1967). One would also expect this approximation to become worse for larger Rayleigh numbers.

However, the above approximation has produced some fairly consistent results for non-rotational Bénard convection. Roberts (1966) found that for large Rayleigh numbers the Nusselt number N (the ratio of heat actually transported across the layer to that which would be conducted if the fluid were immobilized) behaved as $N \sim (R \ln R)^{\frac{1}{5}}$. The asymptotic laminar solution for infinite Prandtl number between rigid–rigid horizontal boundaries has been properly solved by G. Roberts (unpublished), who found $N \sim R^{\frac{1}{5}}$, which is in close agreement with the approximation method. A slight modification of the approximation method gives the preferred mode – that is, the value of the horizontal wavenumber a and the planform of the convection pattern which is 'relatively stable'. This is found to be in the same sense as that of Malkus & Veronis (1958). The approximation method has been applied to convection generated by heat sources (Roberts 1967). Roberts found that the preferred horizontal wavenumber a should increase slightly with increasing Rayleigh number. Initially this did not seem to be borne out by experiment. Tritton & Zarroga (1967) reported that a decreased rapidly with increasing Rayleigh number. Thirlby (1970) attempted to resolve the dilemma by solving the full partial differential equations by computer. He obtained results in good agreement with those of the approximate method. Moreover, in a subsequent series of experiments under Tritton's direction, Hooper (1971) could not repeat the findings of Tritton & Zarroga but found a gradual decrease of a as the Rayleigh number is increased. In further support of the approximation method P. H. Roberts (unpublished) examined the effect of adding an additional term $G(z)\cos 2ax$ to the expansion of the temperature θ and found little effect on the Nusselt number, even for large Rayleigh numbers. However, he only examined large and moderate values of the Prandtl number. The results of the present paper are also in agreement with experimental and numerical results. One may also observe that the approximation leads to equations closely related to the mean field equations often applied to turbulent convection.

For rotational Bénard convection the relevant parameters are the Rayleigh number $R = \alpha g \Delta\theta d/\nu K$, the Taylor number $T = 4d^4\Omega_3^2/\nu^2$ and the Prandtl number $Pr = \nu/K$, where α is the coefficient of volume expansion, g the acceleration due to gravity, $\Delta\theta$ the temperature difference between the horizontal surfaces, ν the kinematic viscosity, K the thermal diffusivity, d the distance between the horizontal boundaries and Ω_3 the rotation about the vertical direction. The dependence of the Rayleigh number and horizontal wavenumber a upon the Taylor number is taken to be $R \sim \rho T^{\alpha'}$ and $a \sim AT^{\beta}$, where α' and β are positive numbers. The asymptotic theory is derived for rolls as the Rayleigh number becomes large. For rolls the convection equations are independent of the Prandtl number, but their solution depends upon it for another reason. R is large when ρ or T are large and corresponding thermal layers develop at the horizontal boundaries. The increase in R due to an increase in ρ can be associated with an increase in the temperature difference $\Delta\theta$ between the horizontal boundaries. The increase in R due to an increase in T is required because of the stabilizing effect of the rotation Ω_3. For a liquid of large Prandtl number such as water $\nu \gg K$. Then since $R \sim O(1/\nu K)$ and $T \sim O(1/\nu^2)$; for a given large $\Delta\theta$ and Ω_3, ρ will be greater than T. For a liquid of small Prandtl number such as mercury $\nu \ll K$, and for a given $\Delta\theta$ and Ω_3, T will be greater than ρ. Thus the order in which ρ or T are allowed to become large is important. When T is greater than ρ the differential equations of the convection can be reduced from eighth order to sixth order because a is also large. In the case of a liquid of small Prandtl number one would expect steady convection to be of smaller magnitude than those in a liquid of large Prandtl number. The asymptotic theories for both limits of ρ and T increasing are dealt with.

For ρ large and T increasing, for rigid–rigid boundaries the ρ thermal layer has the same structure as for non-rotating Bénard convection. The interior of the liquid is isothermal, that is, the temperature gradient $D\theta_0$ is zero. As the Taylor number becomes large the isothermal equation takes the form of a thermal-wind equation in which the vertical gradient of the horizontal zonal velocity v is balanced by the horizontal temperature gradient. The thermal-wind equation is found to have boundary layers of thickness $O[1/a(\ln a)^{\frac{1}{2}}]$. As the Taylor number becomes large three types of thermal layers are possible. They are (1) a linear Ekman layer, (2) a nonlinear Ekman layer and (3) a Blasius-type thermal layer. Case 1 occurs at a higher Taylor number than case 2 and case 2 at a higher one than case 3. Case 2 is only briefly discussed. In case 1 subcritical instability is indicated when A is large and small. The Nusselt number decreases as T is increased for a given R. In case 3 subcritical instability is indicated for large A, but the Nusselt number increases as T is increased for a given R. When the horizontal boundaries are both free the ρ thermal layer is again of the same structure as for the non-rotational problem. As T increases two types of thermal layer are possible: (1) a nonlinear Ekman layer and (2) a Blasius-type thermal layer. Case 1 is not considered. In case 2 no subcritical instability is indicated and the Nusselt number increases as T increases for a given R.

If the Taylor number is large and ρ is increased the convection equations can be reduced from eighth order to sixth since the wavenumber a is large with T. If the vertical velocity $W \ll a$, the equations reduce to the equations for marginal convection or close to marginal convection. If $W \gg a$ the equations have thermal-layer solutions. For rigid–rigid boundaries there are Ekman layers at these boundaries and an isothermal-wind balance in the interior of the liquid. The ρ thermal layers now have a structure different from non-rotating Bénard convection. No subcritical instability is found and the Nusselt number decreases as T is increased for a fixed R. For free–free horizontal boundaries subcritical instability is possible for small A. The Nusselt number decreases as T increases for a given R.

Small amplitude convection ($W \ll a$) close to the marginal case is investigated for free–free boundaries and the results agree with those of Veronis (1959).

2. The convection equations

For a right-handed system of axes OX, OY and OZ the equations of motion for a liquid with rotation $\mathbf{\Omega} = (0, 0, \Omega_3)$ are in tensor form

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0, \tag{2.1}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = X_i - \frac{\partial}{\partial x_i}\left(\frac{P}{\rho} - \tfrac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2\right) + \nu \nabla^2 u + 2\epsilon_{ij3} u_j \Omega_3, \tag{2.2}$$

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = K \nabla^2 \theta, \tag{2.3}$$

where $\mathbf{u}$ is the velocity, $\mathbf{X}$ the external force, P the pressure, ρ the density, $\mathbf{r}$ the radius vector from the origin, ϵ_{ij3} the alternating tensor, ν the kinematic viscosity, K the thermal diffusivity and θ the temperature (see Chandrasekar 1961). Now $X_i = (0, 0, -g)$ and $\rho = \rho_0(1 - \alpha\theta)$, where g is the acceleration of gravity, α the coefficient of volume expansion, and the temperature θ is zero at $z = 0$. Using the Boussinesq approximation, in which the fluid is taken to be incompressible and the change in density is only taken into account in the buoyancy term $\alpha\theta g$, equations (2.1) and (2.2) become

$$\partial u_i / \partial x_i = 0 \tag{2.4}$$

and

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{P}{\rho} - \tfrac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2\right) + \alpha\theta g \lambda_i + \nu \nabla^2 u_i + 2\epsilon_{ij3} u_j \Omega_3, \tag{2.5}$$

where the extra constant $-g$ is absorbed into the first term on the right-hand side in the equation for u_3 and where $\boldsymbol{\lambda} = (0, 0, +1)$. The equations are now put into dimensionless form by letting $\mathbf{x} = \mathbf{x}'d$, $t = t'd$, $\theta = \Delta\theta\theta'$ and $\mathbf{u} = (K/d)\,\mathbf{u}'$, where $\theta = -\Delta\theta$ at $z = d$. The equations of continuity, momentum and energy are, after dropping the primes,

$$\partial u_i / \partial x_i = 0, \tag{2.6}$$

$$\frac{\partial u_i}{\partial t}+\frac{1}{Pr}u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial \overline{w}}{\partial x_i}+R\theta\lambda_i+\nabla^2 u_i+T^{\frac{1}{2}}\epsilon_{ij3}u_j, \tag{2.7}$$

$$Pr\frac{\partial \theta}{\partial t}+u_i\frac{\partial \theta}{\partial x_i} = \nabla^2\theta, \tag{2.8}$$

where $R = \alpha g\Delta\theta d/\nu K$ is the Rayleigh number, $Pr = \nu/K$ is the Prandtl number, $T = 4d^4\Omega_3^2/\nu^2$ is the Taylor number and $\overline{w}$ is the modified pressure. The boundary conditions for a rigid boundary are

$$\mathbf{u} = 0 \text{ at } z = 0, 1; \quad \theta = 0 \text{ at } z = 0; \quad \theta = -1 \text{ at } z = 1.$$

3. The evolutionary criterion of Glansdorff & Prigogine

Following Roberts's (1966) modification of the Glansdorff & Prigogine evolutionary criterion we let

$$\Psi_1 = -(\partial u_i/\partial t)^2, \quad \Psi_2 = -(\partial\theta/\partial t)^2. \tag{3.1}$$

Integrating (2.6)–(2.8) throughout the volume of the liquid gives

$$\int \Psi_1\,d\mathbf{x} = \int\left\{\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right]-\frac{u_iu_j}{Pr}\frac{\partial^2 u_i}{\partial t\,\partial x_j}-R\theta\frac{\partial w}{\partial t}+T^{\frac{1}{2}}\epsilon_{ij3}u_i\frac{\partial u_j}{\partial t}\right\}d\mathbf{x} \tag{3.2}$$

and
$$\int \Psi_2\,d\mathbf{x} = \frac{1}{Pr}\int\left\{\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial \theta}{\partial x_j}\right)^2\right]-u_j\theta\frac{\partial^2\theta}{\partial t\,\partial x_j}\right\}d\mathbf{x}, \tag{3.3}$$

where $u_i = (u, v, w)$.

Keeping close to a stationary state, let $u_j = u_{0j}+\delta u_j$, where

$$\partial u_j/\partial t = \partial(\delta u_j)/\partial t,$$

and $\theta = \theta_0+\delta\theta$, where
$$\partial\theta/\partial t = \partial(\delta\theta)/\partial t.$$

Ignoring first-order and higher-order quantities in δu_j and $\delta\theta$ the integrals (3.2) and (3.3) become

$$\int \Psi_1\,d\mathbf{x} = \int\left\{\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right]-\frac{1}{Pr}u_{0i}u_{0j}\frac{\partial^2 u_i}{\partial t\,\partial x_j}-R\theta_0\frac{\partial w}{\partial t}+T^{\frac{1}{2}}\epsilon_{ij3}u_{0i}\frac{\partial u_j}{\partial t}\right\}d\mathbf{x} = \frac{\partial(\Phi_1)}{\partial t} \tag{3.4}$$

and
$$\int \Psi_2\,d\mathbf{x} = \frac{1}{Pr}\int\left\{\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial \theta}{\partial x_j}\right)^2\right]-u_{0j}\theta_0\frac{\partial^2\theta}{\partial t\,\partial x_j}\right\}d\mathbf{x} = \frac{\partial(\Phi_2)}{\partial t}, \tag{3.5}$$

where
$$\Phi_1 = \int\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}\right)^2-\frac{1}{Pr}u_{0i}u_{0j}\frac{\partial u_i}{\partial x_j}-R\theta_0 w+T^{\frac{1}{2}}\epsilon_{ij3}u_{0i}u_j\right]d\mathbf{x} \tag{3.6}$$

and
$$\Phi_2 = \frac{1}{Pr}\int\left[\frac{1}{2}\left(\frac{\partial\theta}{\partial x_j}\right)^2-u_{0j}\theta_0\frac{\partial\theta}{\partial x_j}\right]d\mathbf{x}. \tag{3.7}$$

Φ_1 and Φ_2 are called the local potentials.

The evolutionary criterion states that since Ψ_1 and Ψ_2 are negative the local potentials Φ_1 and Φ_2 take a minimum value at a stationary state. It can be verified that the variational equations

$$\left(\frac{\delta\Phi_1}{\delta u}\right)_{\mathbf{u}=\mathbf{u}_0,\,\theta=\theta_0} = \left(\frac{\delta\Phi_2}{\delta\theta}\right)_{\mathbf{u}=\mathbf{u}_0,\,\theta=\theta_0} = 0 \tag{3.8}$$

do in fact reduce to the steady-state form of (2.7) and (2.8). For a rotating fluid the velocities can be expressed after dropping the primes as

$$u = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial \zeta}{\partial y}\right), \quad v = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial y \partial z} - \frac{\partial \zeta}{\partial x}\right), \quad w, \tag{3.9}$$

where $\zeta = (\partial v/\partial x) - (\partial u/\partial y)$ is the vertical component of vorticity (Chandrasekar 1961, p. 24). Also $a'^2 = d^2a^2$ and $\zeta = (K/d)\zeta'$. Owing to the periodicity in the x, y plane, w, θ and ζ are expanded as Fourier series. The first few terms are written as

$$w = W(z) f(x,y), \quad \zeta = Z(z) f(x,y), \quad \theta = \theta_0(z) + F(z) f(x,y), \tag{3.10}$$

where $f(x,y)$ is the planform or cell shape in the horizontal plane and satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -a^2 f. \tag{3.11}$$

Thus the velocities become

$$u = \frac{1}{a^2}\left(\frac{\partial f}{\partial x} DW + \frac{\partial f}{\partial y} Z\right), \quad v = \frac{1}{a^2}\left(\frac{\partial f}{\partial y} DW - \frac{\partial f}{\partial x} Z\right), \quad w = Wf, \tag{3.12}$$

where $D = d/dz$.

Equations (3.12) are now substituted into (3.6) and (3.7). Horizontal averages are taken and are denoted by angular brackets. f is normalized so that $\langle f^2 \rangle = 1$. Also $\langle w \rangle = \langle \zeta \rangle = 0$ and $\langle \theta \rangle = \theta_0(z)$ since $\langle f \rangle = 0$. $\frac{1}{2}\langle f^3 \rangle$ is denoted by C. Φ_1 and Φ_2 then become

$$\begin{aligned} a^2\Phi_1 = \int_0^1 \{ & \tfrac{1}{2}[2a^2(DW)^2 + (D^2W)^2 + a^4W^2 + a^2Z^2 + (DZ)^2] - Ra^2F_0 W \\ & + T^{\frac{1}{2}}[Z_0 DW - ZDW_0] - (C/Pr)[\tfrac{1}{2}DW(DW_0)^2 - \tfrac{3}{2}Z_0^2 DW + 2Z_0 ZDW_0 \\ & + W_0 DW_0 D^2W + Z_0 W_0 DZ + a^2 W W_0 DW_0 + 2a^2 W_0^2 DW]\}\, dz \end{aligned} \tag{3.13}$$

and
$$\begin{aligned} Pr\,\Phi_2 = \int_0^1 \{ & \tfrac{1}{2}[(D\theta_0)^2 + (DF)^2 + a^2F^2] \\ & - [\theta_{00} D(W_0 F) + F_0 W_0 D\theta_0 + CF_0(FDW_0 + 2W_0 DF)]\}\, dz. \end{aligned} \tag{3.14}$$

The calculus of variations is now used to minimize Φ_1 and Φ_2 with respect to the functions W, F, θ_0 and Z at the stationary state $W = W_0$, $F = F_0$, $\theta_0 = \theta_{00}$ and $Z = Z_0$. After dropping the suffix zero, Φ_1 gives for the stationary state

$$(D^2 - a^2) Z = -T^{\frac{1}{2}} DW + (C/Pr)(WDZ - ZDW) \tag{3.15}$$

and
$$(D^2 - a^2)^2 W = Ra^2 F + \frac{C}{Pr}[WD(D^2 - a^2) W + 2DW(D^2 - a^2) W] + \frac{3C}{Pr} ZDZ + T^{\frac{1}{2}} DZ. \tag{3.16}$$

$Pr\,\Phi_2$ gives
$$D^2\theta_0 = D(FW) \tag{3.17}$$

and
$$(D^2 - a^2) F = WD\theta_0 + C(FDW + 2WDF). \tag{3.18}$$

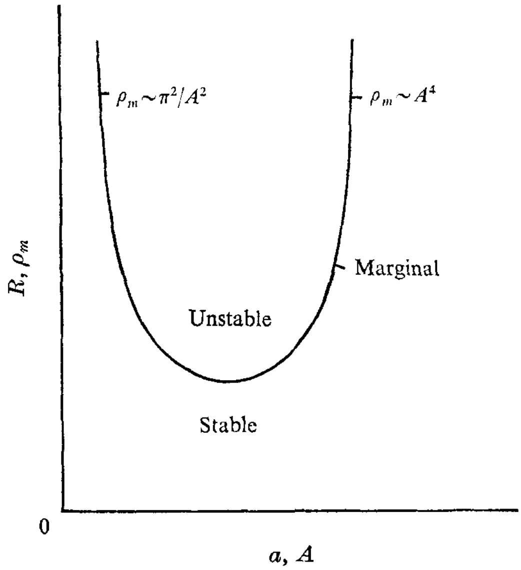

FIGURE 1

For rolls $C = 0$ (Roberts 1966), and (3.15)–(3.18) reduce to

$$D\theta_0 = -N + FW, \tag{3.19}$$

$$N = 1 + \int_{z=0}^{1} FW\, dz, \tag{3.20}$$

$$(D^2 - a^2)F = WD\theta_0, \tag{3.21}$$

$$(D^2 - a^2)Z = -T^{\frac{1}{2}}DW \tag{3.22}$$

and

$$(D^2 - a^2)^2 W = Ra^2F + T^{\frac{1}{2}}DZ, \tag{3.23}$$

where the Nusselt number N is a constant. The boundary conditions are (Chandrasekar 1961, p. 90)

$$W = F = 0 \text{ at } z = 0, 1; \quad \theta = 0 \text{ at } z = 0; \quad \theta = -1 \text{ at } z = 1. \tag{3.24}$$

At a rigid boundary

$$Z = 0, \quad DW = 0. \tag{3.25}$$

At a free boundary

$$DZ = 0, \quad D^2W = 0. \tag{3.26}$$

Eliminating F and Z from the nonlinear equations (3.21)–(3.23) gives

$$(D^2 - a^2)^3 W + TD^2W = Ra^2(D^2 - a^2) F = Ra^2WD\theta_0, \tag{3.27}$$

or

$$(D^2 - a^2)\{W^{-2}[(D^2 - a^2)^3 W + TD^2W + NRa^2W]\} = (D^2 - a^2)^3 W + TD^2W. \tag{3.28}$$

Also, from (3.19) and (3.20),

$$\int_0^1 \frac{1}{W}[(D^2 - a^2)^3 W + TD^2W]\, dz = -Ra^2. \tag{3.29}$$

4. The asymptotic theory for rolls ($R \to \infty$)

For cell shapes which have $C = 0$, when $N = 1$ and $D\theta_0 = -1$ equations (3.19)–(3.23) give for the vertical velocity W

$$(D^2 - a^2)^3 W + TD^2W = Ra^2 W. \tag{4.1}$$

This is the equation that governs the onset of marginal convection. Niiler & Bishopp (1965) solved (4.1) as $T \to \infty$ for rigid–rigid boundaries. They assumed that the exchange of stabilities is true for marginal convection in liquids of large Prandtl number. They found that

$$R \sim \rho_m T^{\frac{2}{3}}, \quad a \sim AT^{\frac{1}{6}} \tag{4.2}$$

and
$$\rho_m A^2 = \pi^2 + A^6 - (2\sqrt{2}/A^2 T^{\frac{1}{12}}) + O(T^{-\frac{1}{6}}). \tag{4.3}$$

Thus as $T \to \infty$ let
$$R \sim \rho T^{\alpha'}, \quad a \sim AT^{\beta}, \tag{4.4}$$

where α' and β are positive constants. Thermal boundary layers develop as R becomes large when ρ or T becomes large. The order in which ρ or T are allowed to increase is important as already stated. For marginal convection when the horizontal boundaries are both free the asymptotic theory as $T \to \infty$ gives

$$R \sim \rho_m T^{\frac{2}{3}}, \quad a \sim AT^{\frac{1}{6}} \tag{4.5}$$

and
$$\rho_m A^2 = \pi^2 + A^6 + O(T^{-\frac{1}{6}}). \tag{4.6}$$

The term of $O(T^{-\frac{1}{12}})$ in (4.3) is the contribution from the Ekman layers.

5. The solution of the convection equations for rigid–rigid boundaries for the limits $\rho \to \infty$, $T \to \infty$

5.1. *The ρ convection*

For R large ($\rho \to \infty$) and taking $D \sim O(1)$ the dominant terms in (3.28) are

$$[D^2 - a^2]^3 W + TD^2W \sim NRa^2(D^2 - a^2)/W. \tag{5.1}$$

This equation could have been obtained from (3.27) by letting $D\theta_0 = 0$, that is, $FW = N$. It is therefore the equation representing isothermal flow. For $\rho \to \infty$ the dominant terms in (5.1) are D^6W and NRa^2D^2/W. A solution at $z = 0$ satisfying these terms and the boundary conditions $W = DW = 0$ at $z = 0$ is

$$W \sim (NRa^2)^{\frac{1}{2}} z^2 (\ln z^{-1})^{\frac{1}{2}}. \tag{5.2}$$

Therefore, setting $z = \epsilon\eta$, where ϵ is the boundary-layer thickness, W becomes

$$W \sim (NRa^2 \ln \epsilon^{-1})^{\frac{1}{2}} \epsilon^2 \left(\eta^2 - \frac{\eta^2 \ln \eta}{2 \ln \epsilon^{-1}} + \dots\right). \tag{5.3}$$

Therefore in the full equations (3.28) let

$$W \sim (NRa^2 \ln \epsilon^{-1})^{\frac{1}{2}} \epsilon^2 \left(f - \frac{g}{\ln \epsilon^{-1}} + \dots\right). \tag{5.4}$$

Taking $\epsilon^4 T \to 0$ and $\epsilon^2 a^2 \to 0$ as $\rho \to \infty$ the equation for f is

$$D^2[f^{-2}(D^6 f + \epsilon^6 NRa^2 f)] = (\epsilon^6 NRa^2 \ln \epsilon^{-1}) D^6 f + O(1/\ln \epsilon^{-1}), \tag{5.5}$$

where $D = d/d\eta$. If $\epsilon^6 N Ra^2 \ln \epsilon^{-1} = 1$ then (5.5) gives

$$D^2(f^{-2}D^6 f) = D^6 f, \tag{5.6}$$

with $f = Df = D^4 f = 0$ at $\eta = 0$ and $f \to \eta^2$ as $n \to \infty$. The boundary condition $D^4 f = 0$ at $\eta = 0$ follows from (3.22) and (3.23).

Integrating (5.6) twice gives

$$D^6 f = f^2 D^4 f + (A + B\eta) f^2.$$

For $f \to \eta^2$ as $\eta \to \infty$, A and B must be taken to be zero. Thus

$$D^6 f = f^2 D^4 f, \tag{5.7}$$

with $f = Df = D^4 f = 0$ at $\eta = 0$ and $f \to \eta^2$ as $\eta \to \infty$.

Since (5.7) has two solutions $\pm(30)^{\frac{1}{2}}/(\eta - \eta_0)$ the given boundary conditions determine a unique solution (Stewartson: see appendix to Roberts (1966)). This solution is

$$f = \eta^2. \tag{5.8}$$

The coefficient of $\ln \epsilon^{-1}$ gives for g

$$D^2[\eta^{-4}(D^6 g - \eta^2)] = D^6 g, \tag{5.9}$$

with $g = Dg = D^4 g = 0$ at $\eta = 0$ and $g \to \frac{1}{2}\eta^2 \ln \eta$ as $\eta \to \infty$.

Integrating (5.9) twice gives

$$D^6 g = \eta^4 D^4 g + \eta^2 + (A + B\eta),$$

where A and B are constants. Since $g \to \frac{1}{2}\eta^2 \ln \eta$ as $\eta \to \infty$, A and B must be taken to be zero. Therefore

$$D^6 g = \eta^4 D^4 g + \eta^2, \tag{5.10}$$

with $g = Dg = D^4 g = 0$ at $\eta = 0$ and $g \to \frac{1}{2}\eta^2 \ln \eta$ as $\eta \to \infty$.

Thus the boundary layer has the same structure as for non-rotating Bénard convection (Roberts 1966).

Integrating (3.29) through these boundary layers gives

$$\frac{k}{\epsilon^5 \ln \epsilon^{-1}} + a^6 \sim Ra^2, \tag{5.11}$$

where $$k = 2\int_0^\infty \frac{D^6 g}{\eta^2} d\eta = 2\int_0^\infty (1 + \eta^2 D^4 g)\, d\eta = 2.221$$

(Stewartson: see appendix to Roberts 1966).

From (5.11) and the relation $\epsilon^6 N Ra^2 \ln \epsilon^{-1} = 1$ it follows that

$$\epsilon \sim 1{\cdot}619[(Ra^2 - a^6)\ln(Ra^2 - a^6)]^{-\frac{1}{5}} \tag{5.12}$$

and $$N \sim 0{\cdot}2782[1 - (a^4/R)]^{\frac{8}{5}}[Ra^2 \ln(Ra^2 - a^6)]^{\frac{1}{5}}. \tag{5.13}$$

5.2. *The T convection*

For T large the wavenumber a becomes large and the dominant terms in (5.1) are

$$TD^2 W \sim NRa^2(D^2 - a^2)/W + a^6 W. \tag{5.14}$$

This is the thermal-wind equation in which the vertical gradient of the horizontal zonal velocity v is balanced by the horizontal temperature gradient. For large a (5.14) can be further reduced to the main-stream equation:

$$TD^2W \sim -(NRa^4/W)+a^6W \tag{5.15}$$

A solution of (5.15) at $z = 0$ satisfying the boundary condition $W = 0$ at $z = 0$ is

$$W \sim (2NRa^4/T)^{\frac{1}{2}} z(\ln z^{-1})^{\frac{1}{2}}. \tag{5.16}$$

Substituting $z = \tau\eta$ into (5.16), where τ is the boundary-layer thickness of the thermal-wind equation (5.14), W becomes

$$W \sim \left(\frac{2NRa^4}{T}\ln\tau^{-1}\right)^{\frac{1}{2}}\tau\left(\eta-\frac{\eta\ln\eta}{2\ln\tau^{-1}}+\ldots\right). \tag{5.17}$$

Therefore in (5.1) or (5.14) let W be

$$W \sim \left(\frac{2NRa^4}{T}\ln\tau^{-1}\right)^{\frac{1}{2}}\tau\left(f-\frac{g}{\ln\tau^{-1}}+\ldots\right), \tag{5.18}$$

where $f \to \eta$ and $g \to \frac{1}{2}\eta\ln\eta$ as $\eta\to\infty$.

This results in

$$\frac{1}{\tau^4T}(D^2-\tau^2a^2)^3f+D^2f \sim \frac{1}{2\tau^2a^2\ln\tau^{-1}}(D^2-\tau^2a^2)\frac{1}{f}. \tag{5.19}$$

Assuming $1/\tau^4T \to 0$ as $T\to\infty$ and letting $2\tau^2a^2\ln\tau^{-1} = 1$, equation (5.19) becomes

$$D^2f = D^2f^{-1}, \tag{5.20}$$

with $f\to\eta$ as $\eta\to\infty$. The solution of (5.20) is

$$f = \tfrac{1}{2}[\eta+(4+\eta^2)^{\frac{1}{2}}], \tag{5.21}$$

giving $f = 1$ at $\eta = 0$.

The terms in $1/\ln\tau^{-1}$ give for g

$$D^2\left[\left(1+\frac{1}{f}\right)g\right] \sim \frac{1}{2f} = \frac{1}{\eta+(4+\eta^2)^{\frac{1}{2}}}, \tag{5.22}$$

with $g\to\frac{1}{2}\eta\ln\eta$ as $\eta\to\infty$. If (5.22) is integrated twice g is given by

$$g = \frac{f}{1+f}\left\{\tfrac{1}{2}\eta\ln\left[\frac{\eta+(4+\eta^2)^{\frac{1}{2}}}{2}\right]+\tfrac{1}{2}[\eta-(4+\eta^2)^{\frac{1}{2}}]+\tfrac{1}{24}[(4+\eta^2)^{\frac{3}{2}}-\eta^3]+\tfrac{2}{3}\right\}, \tag{5.23}$$

which gives $g = 0$ at $\eta = 0$.

The thickness of the thermal-wind boundary layer is

$$\tau = 1/a(\ln a)^{\frac{1}{2}}. \tag{5.24}$$

From (5.18) using $2\tau^2a^2\ln\tau^{-1} = 1$, W in the boundary layer can be written as

$$W \sim \left(\frac{NRa^2}{T}\right)^{\frac{1}{2}}\left(f-\frac{g}{\ln\tau^{-1}}+\ldots\right). \tag{5.25}$$

Thus in the full equations (3.28) let W be

$$W \sim \left(\frac{NRa^2}{T}\right)^{\frac{1}{2}}\left(f-\frac{g}{\ln\sigma^{-1}}+\ldots\right), \tag{5.26}$$

where σ is the thickness of the thermal boundary layer of equation (3.28). Both τ and σ are functions of T and thus a matching can be carried out between (5.25) and (5.26). Thus in the full equation (3.28)

$$D^2\left[\frac{1}{f^2}(D^6f+\sigma^4TD^2f+\sigma^6NRa^2f)\right] = \frac{NRa^2\sigma^2}{T}(D^6f+\sigma^4TD^2f)+O\left(\frac{1}{\ln\sigma^{-1}}\right), \quad (5.27)$$

where σ^2a^2 is taken to be small as $T\to\infty$.

Three possibilities now present themselves.

(1) A linear Ekman layer in which

$$\left.\begin{aligned} &\sigma^6NRa^2\ln\sigma^{-1} = 1, \quad \sigma^4T = 4; \\ \text{therefore} \quad &\sigma^2NRa^2/T = 1/4\ln\sigma^{-1}. \end{aligned}\right\} \quad (5.28)$$

(2) A nonlinear Ekman layer in which all terms are of the same order:

$$\left.\begin{aligned} &\sigma^6NRa^2 = 1, \quad \sigma^4T = 4; \\ \text{therefore} \quad &\sigma^2NRa^2/T = \tfrac{1}{4}. \end{aligned}\right\} \quad (5.29)$$

(3) A Blasius-type boundary layer in which

$$\left.\begin{aligned} &\sigma^6NRa^2\ln\sigma^{-1} = 1, \quad NRa^2\sigma^2/T = 1; \\ \text{therefore} \quad &\sigma^4T = 1/\ln\sigma^{-1}. \end{aligned}\right\} \quad (5.30)$$

If T_1, T_2, and T_3 are the Taylor numbers for each case, from (5.28), (5.29) and (5.30),

$$NRa^2 \sim T_1^{\frac{2}{3}}/\ln T_1, \quad NRa^2 \sim T_2^{\frac{2}{3}}, \quad NRa^2 \sim T_3^{\frac{2}{3}}(\ln T_3)^{\frac{1}{2}}. \quad (5.31)$$

For a given NRa^2 (5.31) gives $\quad T_1 > T_2 > T_3. \quad (5.32)$

Case 1. *The linear Ekman layer.* From (5.29) and (5.27)

$$D^2\left[\frac{1}{f^2}(D^6f+4D^2f)+\frac{1}{f\ln\sigma^{-1}}\right] = \frac{1}{4\ln\sigma^{-1}}(D^6f+4D^2f)+O\left(\frac{1}{\ln\sigma^{-1}}\right) \quad (5.33)$$

Thus the equation for f is

$$D^2[f^{-2}(D^6f+4D^2f)] = 0, \quad (5.34)$$

with $f = Df = D^6f+4D^2f = 0$ at $\eta = 0$ and $f\to 1$ as $\eta\to\infty$.

Integrating (5.34) twice gives

$$D^6f+4D^2f = f^2(A+B\eta).$$

Since $f\to 1$ as $\eta\to\infty$, A and B must be taken to be zero. Thus

$$D^6f+4D^2f = 0, \quad (5.35)$$

with $f = Df = D^6f+4D^2f = 0$ at $\eta = 0$ and $f\to 1$ as $\eta\to\infty$.

The solution to (5.35) is

$$f = 1-e^{-\eta}[\cos\eta+\sin\eta]. \quad (5.36)$$

From the terms in $1/\ln\sigma^{-1}$ is obtained

$$D^2\left(\frac{D^6g+4D^2g}{f^2}-\frac{1}{f}\right) = 0, \quad (5.37)$$

with $g = Dg = D^6g+4D^2g = 0$ at $\eta = 0$ and $g\to 0$ as $\eta\to\infty$.

Integrating (5.37) twice results in

$$(D^6g+4D^2g)/f = 1+(A\eta+B)\,[1-e^{-\eta}(\cos\eta+\sin\eta)]. \tag{5.38}$$

When integrating (3.29) the integral

$$\int_{\eta=0}^{\infty}\frac{D^6g+4D^2g}{f}\,d\eta$$

is required. For it to have a finite value A must be taken equal to zero and $B=-1$. Its value is then

$$\int_{\eta=0}^{\infty}e^{-\eta}(\cos\eta+\sin\eta)\,d\eta = 1.$$

Substituting (5.26) into (3.29) and integrating through the ρ thermal layers and the Ekman layers gives

$$\frac{2\cdot2214}{\epsilon^5\ln\epsilon^{-1}}+\frac{2^{\frac{1}{2}}T^{\frac{5}{4}}}{\ln T}+a^6 \sim Ra^2. \tag{5.39}$$

From (5.39) and the relation $\epsilon^6NRa^2\ln\epsilon^{-1}=1$ results

$$\epsilon \sim 1\cdot619\left[\left(Ra^2-a^6-\frac{2^{\frac{1}{2}}T^{\frac{5}{4}}}{\ln T}\right)\ln\left(Ra^2-a^6-\frac{2^{\frac{1}{2}}T^{\frac{5}{4}}}{\ln T}\right)\right]^{-\frac{1}{5}} \tag{5.40}$$

and
$$N \sim 0\cdot2782\left(1-\frac{a^4}{R}-\frac{2^{\frac{1}{2}}T^{\frac{5}{4}}}{Ra^2\ln T}\right)^{\frac{6}{5}}\left[Ra^2\ln\left(Ra^2-a^6-\frac{2^{\frac{1}{2}}T^{\frac{5}{4}}}{\ln T}\right)\right]^{\frac{1}{5}} \tag{5.41}$$

As ϵ is to be positive, (5.40) gives

$$Ra^2 \geqslant O(T^{\frac{5}{4}}/\ln T), \quad R \geqslant O(a^4). \tag{5.42}$$

Thus
$$R \geqslant O[T^{\frac{5}{6}}/(\ln T)^{\frac{2}{3}}], \quad a \leqslant O[T^{\frac{5}{24}}/(\ln T)^{\frac{1}{6}}]. \tag{5.43}$$

From (5.43), $\sigma^2a^2 \leqslant O[1/T^{\frac{1}{12}}(\ln T)^{\frac{1}{3}}]$ and $1/\tau^4T \leqslant O[(\ln T)^{\frac{2}{3}}/T^{\frac{1}{6}}]$ justifying the assumptions made in § 5.2. The contribution to the integral (3.29) from the boundary layer (τ) of the thermal-wind equations is

$$\frac{2T}{\tau}\int_0^{\infty}\frac{D^2f}{f}\,d\eta \leqslant O[T^{\frac{29}{24}}(\ln T)^{\frac{1}{3}}], \tag{5.44}$$

which can be neglected in comparison with terms of $O(T^{\frac{5}{4}}/\ln T)$. In the main stream of the thermal-wind equations $W \sim O[a/\sigma(\ln\sigma^{-1})^{\frac{1}{2}}]$ and in the Ekman layer $W \sim O[1/\sigma(\ln\sigma^{-1})^{\frac{1}{2}}]$. As $\rho\to\infty$, in the main stream $W \sim O[1/\epsilon^3(\ln\epsilon^{-1})^{\frac{1}{2}}]$ and in the ρ thermal layer $W \sim O(1/\epsilon)$.

For $Ra^2 \sim a^6+(2^{\frac{1}{2}}T^{\frac{5}{4}}/\ln T)$, the ρ thermal layer is much thicker than the Ekman layer and the ρ convection tends to zero. However, the T convection still exists. In this case $R = O[T^{\frac{5}{6}}/(\ln T)^{\frac{2}{3}}]$ and $a = O[T^{\frac{5}{24}}/(\ln T)^{\frac{1}{6}}]$, and

$$\rho A^2 \sim A^6+\sqrt{2}. \tag{5.45}$$

Thus, comparing with the marginal stability equation (4.3) gives, for large A, $\rho \sim A^4$ and $\rho_m \sim A^4$; for small A, $\rho \sim \sqrt{2}/A^2$ and $\rho_m \sim \pi^2/A^2$. Thus for $T\to\infty$ ($R\to\infty$) convection exists for values of ρ which are equal to or less than the values for the marginal case. This suggests the existence of subcritical instabilities. Notice that, for A small but fixed, a must become large as $T \to \infty$. Rossby (1969) in his experiments found subcritical instabilities for large Taylor numbers. The physical reason he gives for this is that, when Ekman layers have formed, the bulk of the fluid sees free–free boundaries, which are less constraining according to linear theory. Notice that the Ekman-layer contribution to (5.45) affects the convection of long wavelength (A small).

From (3.22) the vertical vorticity Z in the Ekman layer is

$$Z \sim (T/4 \ln \sigma^{-1})^{\frac{1}{2}} h, \tag{5.46}$$

where $D^2 h = -Df$. Since $F = 0$ at $z = 0$, equation (3.23) gives $D^4 f = 4Dh$ at $z = 0$. From (3.22) and (5.36)

$$h = 1 - e^{-\eta} \cos \eta. \tag{5.47}$$

Case 2. *The nonlinear Ekman layer.* From (5.27) and (5.29),

$$D^2 \left[\frac{1}{f^2} (D^6 f + 4D^2 f + f) \right] = \tfrac{1}{4}[D^6 f + 4D^2 f] + O\left(\frac{1}{\ln \sigma^{-1}}\right), \tag{5.48}$$

with $f = Df = D^6 f + 4D^2 f = 0$ at $\eta = 0$ and $f \to 1$ as $\eta \to \infty$. Integrating (5.48) twice results in

$$D^6 f + 4D^2 f + f = \tfrac{1}{4} f^2 (Df + 4f) + f^2 (A + B\eta).$$

Since $f \to 1$ as $\eta \to \infty$, A and B must be taken to be zero. Therefore the equation for f is

$$D^6 f + 4D^2 f + f = \tfrac{1}{4} f^2 (D^4 f + 4f), \tag{5.49}$$

with $f = Df = D^6 f + 4D^2 f = 0$ at $\eta = 0$ and $f \to 1$ as $\eta \to \infty$.

Equation (5.49) possesses two solutions $f = A/A$ and $f = -B/B$. If these are excluded the four given boundary conditions determine a unique solution to (5.49). Integrating (3.29) through the ρ thermal layers and the nonlinear Ekman layers gives

$$\frac{2{\cdot}2214}{\epsilon^5 \ln \epsilon^{-1}} \sim Ra^2 - a^6 + \frac{k}{2\sqrt{2}} T^{\frac{5}{4}}, \tag{5.50}$$

where
$$k = \int_{\eta=0}^{\infty} \frac{D^6 f + 4D^2 f}{f} d\eta = \frac{1}{4} \int_0^{\infty} [D^2 f]^2 d\eta + \int_0^{\infty} [f^2 - 1] \, d\eta.$$

k can be found from numerical solution of (5.49). From (5.50) and the relation $\epsilon^6 N Ra^2 \ln \epsilon^{-1} = 1$,

$$\epsilon \sim 1{\cdot}619 \left[\left(Ra^2 - a^6 + \frac{k}{2\sqrt{2}} T^{\frac{5}{4}} \right) \ln \left(Ra^2 - a^6 + \frac{k}{2\sqrt{2}} T^{\frac{5}{4}} \right) \right]^{-\frac{1}{5}} \tag{5.51}$$

and
$$N \sim 0{\cdot}2782 \left(1 - \frac{a^4}{R} + \frac{k}{2\sqrt{2}} \frac{T^{\frac{5}{4}}}{Ra^2} \right)^{\frac{6}{5}} \left[Ra^2 \ln \left(Ra^2 - a^6 + \frac{k}{2\sqrt{2}} T^{\frac{5}{4}} \right) \right]^{\frac{1}{5}} \tag{5.52}$$

It is not proposed to take this case any further as it is similar to case 1 or 3.

Case 3. *The Blasius-type thermal layer.* As was pointed out in (5.32), this thermal layer occurs at a lower Taylor number than does the Ekman layer. From (5.27) and (5.30),

$$D^2 \left[\frac{1}{f^2} \left(D^6 f + \frac{D^2 f}{\ln \sigma^{-1}} + \frac{f}{\ln \sigma^{-1}} \right) \right] = \left(D^6 f + \frac{D^2 f}{\ln \sigma^{-1}} \right) + O\left(\frac{1}{\ln \sigma^{-1}}\right). \tag{5.53}$$

From (3.22) the vertical vorticity Z in the thermal layer is given by $Z \sim T^{\frac{1}{2}}h$, where $D^2h = -Df$. Since $F = 0$ at $z = 0$ equation (3.23) gives $D^4f \sim Dh/\ln\sigma^{-1}$, which tends to zero as $T \to \infty$. Thus from (5.53) we have

$$D^2(f^{-2}D^6f) = D^6f, \tag{5.54}$$

with $f = Df = D^4f = 0$ at $\eta = 0$ and $f \to 1$ as $\eta \to \infty$.

Integrating (5.54) twice gives

$$D^6f = f^2Df + f^2(A + B\eta).$$

Since $f \to 1$ as $\eta \to \infty$, A and B must be zero. Thus the equation for f is

$$D^6f = f^2D^4f, \tag{5.55}$$

with $f = Df = D^4f = 0$ at $\eta = 0$ and $f \to 1$ as $\eta \to \infty$.

Since (5.55) has two singular solutions $f = \pm(30)^{\frac{1}{2}}/(\eta - \eta_0)$ containing two arbitrary constants, the four given boundary conditions determine a unique solution. Substituting (5.26) into the integral (3.29) and taking into consideration the ρ thermal layer and the Blasius-type thermal layer yields

$$\frac{2{\cdot}2214}{\epsilon^5 \ln\epsilon^{-1}} \sim Ra^2 - a^6 + \frac{k}{2\sqrt{2}}(T\ln T)^{\frac{5}{4}}, \tag{5.56}$$

where

$$k = \int_0^\infty \frac{D^6f}{f}\,d\eta = \int_0^\infty (D^2f)^2\,d\eta.$$

Therefore k is a positive number. From (5.56) and the relation $\epsilon^6 N Ra^2 \ln\epsilon^{-1} = 1$ we have

$$\epsilon \sim 1{\cdot}619\left\{\left[Ra^2 - a^6 + \frac{k}{2\sqrt{2}}(T\ln T)^{\frac{5}{4}}\right]\ln\left[Ra^2 - a^6 + \frac{k}{2\sqrt{2}}(T\ln T)^{\frac{5}{4}}\right]\right\}^{-\frac{1}{5}} \tag{5.57}$$

and

$$N \sim 0{\cdot}2782\left[1 - \frac{a^4}{R} + \frac{k}{2\sqrt{2}}\frac{(T\ln T)^{\frac{5}{4}}}{Ra^2}\right]^{\frac{6}{5}}\left[Ra^2\ln\left(Ra^2 - a^6 + \frac{k}{2\sqrt{2}}(T\ln T)^{\frac{5}{4}}\right)\right]^{\frac{1}{5}}. \tag{5.58}$$

From (5.57), for ϵ to be positive

$$R \geqslant O(a^4), \quad a \leqslant O[(T\ln T)^{\frac{5}{24}}]. \tag{5.59}$$

Thus,

$$R \geqslant O[(T\ln T)^{\frac{5}{6}}], \quad a \leqslant O[(T\ln T)^{\frac{5}{24}}]. \tag{5.60}$$

Also $\sigma^2a^2 \leqslant O[(\ln T)^{\frac{5}{12}}/T^{\frac{1}{12}}]$ and $1/\tau^4T < O[(\ln T)^{\frac{17}{6}}/T^{\frac{1}{6}}]$ and tend to zero as $T \to \infty$. The contribution from the thermal-wind boundary layer is

$$\frac{T}{\tau}\int_0^\infty \frac{D^2f}{f}\,d\eta \leqslant O[T^{\frac{29}{24}}(\ln T)^{\frac{17}{24}}], \tag{5.61}$$

and this can be neglected in comparison with terms of $O[(T\ln T)^{\frac{5}{4}}]$. In the main stream of the thermal-wind equation $W \sim O[a/\sigma(\ln\sigma^{-1})^{\frac{1}{2}}]$ and in the Blasius-type thermal layer $W \sim O[1/\sigma(\ln\sigma^{-1})^{\frac{1}{2}}]$, where $\sigma \sim 2^{\frac{1}{2}}[T\ln T]^{-\frac{1}{4}}$. For $\rho \to \infty$, in the main stream $W \sim O[1/\epsilon^3(\ln\epsilon^{-1})^{\frac{1}{2}}]$ and $W \sim O(1/\epsilon)$ in the ρ thermal layer. The ρ thermal layer is much thicker than the Blasius layer and the ρ convection tends to zero for

$$Ra^2 \sim a^6 - (k/2\sqrt{2})\,[T\ln T]^{\frac{5}{4}}, \tag{5.62}$$

i.e.

$$\rho A^2 \sim A^6 - (k/2\sqrt{2}). \tag{5.63}$$

In this case ρ is not defined for small A. Subcritical instability is indicated for large A. From (5.58), for a given R the Nusselt number increases as the Taylor number increases. However, in the linear Ekman layer, from (5.41), the Nusselt number decreases as the Taylor number increases. This agrees with Rossby's (1969) observations that the Nusselt number reaches a maximum as the Taylor number increases for a given Rayleigh number.

6. The solution of the convection equations for free–free boundaries for the limits $\rho \to \infty$, $T \to \infty$

6.1. *The ρ convection*

For $\rho \to \infty$ equation (5.1) becomes

$$(D^2 - a^2)^3 W \sim NRa^2(D^2 - a^2)/W, \tag{6.1}$$

or

$$(D^2 - a^2)^2 W \sim NRa^2/W. \tag{6.2}$$

A solution at $z = 0$ satisfying the boundary conditions $W = D^2W = 0$ at $z = 0$ is

$$W \sim (NRA^2)^{\frac{1}{2}}\left(cz - \frac{z^3}{6c}\ln\frac{1}{z} + \ldots\right), \tag{6.3}$$

where c is a constant. c is equal to the value of DW at $z = 0$ and is independent of the wavenumber a, since at $z = 0$ the terms a^4W and a^2D^2W are zero owing to the free-surface boundary conditions. Also, from (6.2) c is independent of NRa^2, and its value can be determined from the numerical solution of

$$(D^2 - 1)^2 w = 1/w, \tag{6.4}$$

with $w = Dw = 0$ at $z = 0$ and $z = 1$ (P. H. Roberts, unpublished manuscript).

Letting $z = \epsilon\eta$ in (6.3), where ϵ is the boundary-layer thickness, gives

$$W \sim (NRa^2)^{\frac{1}{2}}\,\epsilon\left(c\eta - \frac{\eta^3}{6c}\epsilon^2\ln\frac{1}{\epsilon} + \frac{\eta^3\ln\eta}{6c}\epsilon^2 + \ldots\right). \tag{6.5}$$

Therefore in the full equations (3.28) let W be

$$W \sim (NRa^2)^{\frac{1}{2}}\,\epsilon[f - p\epsilon^2\ln\epsilon^{-1} + (g/c)\,\epsilon^2 + \ldots], \tag{6.6}$$

with $f \to c\eta$, $p \to \eta^3/6c$ and $g \to \frac{1}{6}\eta^3\ln\eta$ as $\eta \to \infty$.

Thus from (3.28)

$$D^2[f^{-2}(D^6f + \epsilon^6NRa^2f)] = \epsilon^4NRa^2D^6f + O(\epsilon^2\ln\epsilon^{-1}). \tag{6.7}$$

Taking $\epsilon^4NRa^2 = 1$ the equation for f is

$$D^2(f^{-2}D^6f) = D^6f, \tag{6.8}$$

with $f = D^2f = D^4f = 0$ at $\eta = 0$ and $f \to c\eta$ as $\eta \to \infty$. Notice that $D^4f = 0$ at $\eta = 0$ follows from (3.23) since $F = DZ = 0$ at $z = 0$. Integrating (6.7) twice we have

$$D^6f = f^2D^4f + f^2(A + B\eta).$$

Since $f \to c\eta$ as $\eta \to \infty$, A and B must be zero. The equation for f is then

$$D^6f = f^2D^4f, \tag{6.9}$$

with $f = D^2f = D^4f = 0$ at $= 0$ and $f \to c\eta$ as $\eta \to \infty$.

Since (6.9) has two solutions $f = \pm(30)^{\frac{1}{2}}/(\eta-\eta_0)$, the given four boundary conditions determine a unique solution to (6.9):

$$f = c\eta. \tag{6.10}$$

The terms in $\epsilon^2 \ln \epsilon^{-1}$ give for p

$$D^2(f^{-2}D^6p) = D^6p, \tag{6.11}$$

with $p = D^2p = D^4p = 0$ at $\eta = 0$ and $p \to \eta^3/6c$ as $\eta \to \infty$.

Integrating (6.11) twice gives the unique solution

$$p = \eta^3/6c. \tag{6.12}$$

From the terms in ϵ^2 the equation for g is

$$D^2\left[\frac{1}{f^2}\left(\frac{D^6g}{c}+f\right)\right] = \frac{D^6g}{c}, \tag{6.13}$$

with $g = D^2g = D^4g = 0$ at $\eta = 0$ and $g \to \frac{1}{6}\eta^3 \ln \eta$ as $\eta \to \infty$.

Integrating (6.13) twice and using (6.10) gives

$$D^6g/c^2 = \eta^2D^4g-\eta+\eta^2(A+B\eta).$$

Since $g \to \frac{1}{6}\eta^3 \ln \eta$ as $\eta \to \infty$, A and B must be zero. The equation for g is then

$$D^6g/c^2 = \eta^2D^4g-\eta, \tag{6.14}$$

with $g = D^2g = D^4g = 0$ at $\eta = 0$ and $g \to \frac{1}{6}\eta^3 \ln \eta$ as $\eta \to \infty$. The boundary layer has the same structure as for non-rotational Bénard convection. The contribution from the ρ thermal layer to the integral (3.29) is

$$\frac{2}{\epsilon^3c^2}\int_0^\infty \frac{D^6g}{\eta}\,d\eta, \tag{6.15}$$

where
$$\frac{2}{c^2}\int_0^\infty \frac{D^6g}{\eta}\,d\eta = -2\int_0^\infty (1-\eta D^4g)\,d\eta = -\frac{2{\cdot}12365}{c^{\frac{1}{2}}} \tag{6.16}$$

(P. H. Roberts, unpublished manuscript).

6.2. *The T convection*

There are three possibilities arising from (5.27).

(1) A linear Ekman layer in which

$$\left.\begin{aligned} \sigma^4NRa^2 = 1, \quad \sigma^4T = 4;\\ \text{therefore} \qquad \sigma^2NRa^2/T = \tfrac{1}{4}\sigma^2. \end{aligned}\right\} \tag{6.17}$$

Also $\sigma^6NRa^2 = \sigma^2$. The boundary conditions at $\eta = 0$ cannot be satisfied and there is no contribution to the integral (3.29). This case is thus of no interest.

(2) A nonlinear Ekman layer in which

$$\left.\begin{aligned} \sigma^6NRa^2 = 1, \quad \sigma^4T = 4;\\ \text{therefore} \qquad \sigma^2NRa^2/T = \tfrac{1}{4}. \end{aligned}\right\} \tag{6.18}$$

The equation for f is (5.49) with the boundary conditions $f = D^2f = D^4f = 0$ at $\eta = 0$ and $f \to 1$ as $\eta \to \infty$. This case will not be considered in this paper.

(3) A Blasius-type thermal layer in which

$$\left.\begin{aligned} \sigma^4 N R a^2 = 1, \quad N R a^2 \sigma^2 / T = 1; \\ \text{therefore} \qquad \sigma^4 T = \sigma^2. \end{aligned}\right\} \tag{6.19}$$

The Blasius-type thermal layer. From (5.27) and (6.19), ignoring terms of order σ^2 and $\sigma^2 a^2$, the equation for f is

$$D^2(f^{-2} D^6 f) = D^6 f, \tag{6.20}$$

with $f = D^2 f = D^4 f = 0$ at $\eta = 0$ and $f \to 1$ as $\eta \to \infty$.

Integrating (6.20) twice gives

$$D^6 f = f^2 D^4 f + f^2 (A + B\eta).$$

Since $f \to 1$ as $\eta \to \infty$, A and B must be zero. Thus the equation for f is

$$D^6 f = f^2 D^4 f, \tag{6.21}$$

with $f = D^2 f = D^4 f = 0$ at $\eta = 0$ and $f \to 1$ as $\eta \to \infty$. Again (6.21) has a unique solution for the given boundary conditions. Substituting (5.26) into the integral (3.29) and integrating through the ρ thermal layer and the Blasius-type thermal layer gives

$$2{\cdot}21365/\epsilon^3 c^2 \sim R a^2 - a^6 + 2kT^{\frac{5}{2}}, \tag{6.22}$$

where

$$k = \int_0^\infty \frac{D^6 f}{f} d\eta = \int_0^\infty (D^2 f)^2 d\eta$$

is a positive number. For $c = 1$, equation (6.22) and $\epsilon^4 N R a^2 = 1$ gives

$$\epsilon \sim 1{\cdot}285(R a^2 - a^6 + 2kT^{\frac{5}{2}})^{-\frac{1}{3}} \tag{6.23}$$

and

$$N \sim 0{\cdot}3663(R a^2 - a^6 + 2kT^{\frac{5}{2}})^{\frac{1}{3}}. \tag{6.24}$$

From (6.23), for ϵ to be positive,

$$R \geqslant O(a^4), \quad a^6 \leqslant O(T^{\frac{5}{2}}). \tag{6.25}$$

Thus

$$R \geqslant O(T^{\frac{5}{3}}), \quad a \leqslant O(T^{\frac{5}{12}}). \tag{6.26}$$

Therefore $\sigma^2 a^2 \leqslant O(T^{-\frac{1}{6}})$ and $1/\tau^4 T \leqslant O[T^{\frac{2}{3}}(\ln T)^2]$. From the last inequality, for $1/\tau^4 T$ to be negligible as assumed in (5.19) $a \ll O(T^{\frac{5}{12}})$. The contribution from the thermal-wind boundary layer (τ) is of $O(T/\tau) < O[T^{\frac{17}{12}}(\ln T)^{\frac{1}{12}}]$ and can be ignored in comparison with terms of $O(T^{\frac{5}{2}})$. In the thermal-wind main stream $W \sim O(a/\sigma)$ and in the Blasius-type thermal layer $W \sim O(1/\sigma)$, where $\sigma \sim O(T^{-\frac{1}{2}})$. As $\rho \to \infty$ in the main stream $W \sim O(1/\epsilon^2)$ and in the thermal layer $W \sim O(1/\epsilon)$. Since $a \ll O(T^{\frac{5}{12}})$ there is no possibility of the ρ thermal layer being thicker than the Blasius layer. That is, no such relation as $R a^2 \sim a^6 - 2kT^{\frac{5}{2}}$ is possible. In this case we conclude there are no subcritical instabilities. From (6.24) the Nusselt number increases as the Taylor number is increased for a given Rayleigh number.

7. The solution of the convection equations in the limits $T \to \infty$ and $\rho \to \infty$

From (3.21) and (3.22), since the wavenumber a becomes large as $T \to \infty$, and taking D^2F and D^2Z to be $O(1)$ in the bulk of the fluid, we obtain

$$-a^2F \sim WD\theta_0 \tag{7.1}$$

and

$$a^2Z \sim T^{\frac{1}{2}}DW. \tag{7.2}$$

From (7.1) and (3.19) it follows that

$$F \sim NW/(a^2+W^2) \tag{7.3}$$

and

$$D\theta_0 \sim -Na^2/(a^2+W^2). \tag{7.4}$$

From (7.4) and (3.27) we have

$$[D^2-a^2]^3W+TD^2W \sim -[NRa^4W/(a^2+W^2)] \tag{7.5}$$

or

$$(D^2-a^2)^3W+TD^2W+NRa^2W \sim -W^2a^{-2}[(D^2-a^2)^3W+TD^2W]. \tag{7.6}$$

The boundary conditions are

$$W = DW = Z = 0 \quad \text{at} \quad z = 0, 1 \quad \text{for a rigid boundary surface}$$

and

$$W = D^2W = DZ = 0 \quad \text{at} \quad z = 0, 1 \quad \text{for a free surface.}$$

F is zero when W is zero. Assuming $W \gg a$ – that is, finite amplitudes – (7.5) reduces to

$$(D^2-a^2)^3W+TD^2W \sim -(NRa^4/W). \tag{7.7}$$

Equation (7.7) can be obtained from (3.27) by ignoring the term D^2F in comparison with a^2F and taking $D\theta_0 \equiv 0$, i.e. $FW \equiv N$. It is therefore the isothermal equation. If $W \ll a$ – that is, infinitesimal amplitudes – (7.5) reduces to the marginal equation (4.1).

7.1. *Rigid–rigid boundaries*

The T convection (the Ekman layer). As $T \to \infty$, equation (7.5) reduces to the thermal-wind equation

$$TD^2W \sim -\frac{NRa^4W}{a^2+W^2}+a^6W, \tag{7.8}$$

or

$$D^2W+\frac{NRa^2}{T}W \sim -\frac{W^2}{a^2}D^2W+\frac{a^6}{T}W\left(1+\frac{W^2}{a^2}\right). \tag{7.9}$$

For $W \gg a$ equation (7.8) becomes the thermal-wind main-stream equation (5.15). Using (5.16) and (5.18), let W in the thermal-wind boundary layer be

$$W \sim \left(\frac{2NRa^4}{T}\ln\sigma^{-1}\right)^{\frac{1}{2}}\sigma\left(f-\frac{g}{\ln\sigma^{-1}}+\ldots\right), \tag{7.10}$$

where σ is the boundary-layer thickness and $f \to \eta$ and $g \to \frac{1}{2}\eta\ln\eta$ as $\eta \to \infty$. Substituting (7.10) into (7.9) gives

$$D^2f+\frac{\sigma^2NRa^2}{T}f = -\frac{2\sigma^2NRa^2}{T}\ln(\sigma^{-1})f^2D^2f+O\left(\frac{1}{\ln\sigma^{-1}}\right), \tag{7.11}$$

where it is assumed that $\sigma^2 a^6/T \to 0$ as $T \to \infty$. Letting $(2\sigma^2 NRa^2/T)\ln\sigma^{-1} = 1$ gives for f

$$D^2 f = -f^2 D^2 f, \tag{7.12}$$

with $f = 0$ at $\eta = 0$ and $f \to \eta$ as $\eta \to \infty$.

The solution of (7.12) is

$$f = \eta. \tag{7.13}$$

The condition $Df = 0$ at $z = 0$ has to be satisfied in the Ekman layers of equation (7.5). From the terms in $\ln\sigma^{-1}$ the equation for g is

$$D^2 g = f/2(1+f^2) = \eta/2(1+\eta^2), \tag{7.14}$$

with $g = 0$ at $\eta = 0$ and $g \to \frac{1}{2}\eta\ln\eta$ as $\eta \to \infty$.

Integrating (7.14) twice gives for g

$$g = \tfrac{1}{4}\eta\ln(1+\eta^2) + \tfrac{1}{2}\tan^{-1}\eta. \tag{7.15}$$

Now for $T \to \infty$ the main-stream equation of (7.5) is again the thermal-wind main-stream equation (5.15). Substituting (7.10) or (5.18) into (7.6) and denoting the Ekman-layer thickness by σ (it will be found to have the same thickness as the thermal-wind boundary layer) gives

$$D^6 f + 4D^2 f + \sigma^6 NRa^2 f = -\frac{2\sigma^2 NRa^2}{T}\ln(\sigma^{-1}) f^2 (D^6 f + 4D^2 f) + O\left(\frac{1}{\ln\sigma^{-1}}\right). \tag{7.16}$$

Taking $(2\sigma^2 NRa^2/T)\ln\sigma^{-1} = 1$ gives $\sigma^6 NRa^2 \ln\sigma^{-1} = 2$. Also notice that the Ekman layer and thermal-wind boundary layer have the same thickness. The equation for f is

$$(1+f^2)(D^6 f + 4D^2 f) = 0, \tag{7.17}$$

with $f = Df = D^6 f + 4D^2 f = 0$ at $\eta = 0$ and $f \to \eta$ as $n \to \infty$.

The solution of (7.17) is

$$f = \eta + A + e^{-\eta}(-A\cos\eta + B\sin\eta). \tag{7.18}$$

The vertical vorticity Z in the thermal-wind boundary layer (σ) is, from (7.2) and (7.10),

$$Z = (2NR\ln\sigma^{-1})^{\frac{1}{2}}\left[1 - \frac{2+\ln(1+\eta^2)}{4\ln\sigma^{-1}} + \dots\right]. \tag{7.19}$$

In the Ekman layer let Z be given by

$$Z = (2NR\ln\sigma^{-1})^{\frac{1}{2}}\left(h - \frac{p}{\ln\sigma^{-1}} + \dots\right), \tag{7.20}$$

where $h \to 1$ and $p \to \frac{1}{4}[2+\ln(1+\eta^2)]$ as $\eta \to \infty$.

Since $F = 0$ at $z = 0$, equations (3.23) and (7.20) give $\sigma^2 a^2 D^4 f = Dh$ at $\eta = 0$. Thus as $T \to \infty$, $Dh = 0$ at $\eta = 0$, which is the same boundary condition on the vertical vorticity as for a free horizontal boundary. Also (7.2) and (7.18) give for h

$$h = Df = 1 + e^{-\eta}[(A+B)\cos\eta + (A-B)\sin\eta].$$

Since $h = 0$ at $\eta = 0$, $A+B = -1$. Then $Dh = 0$ at $\eta = 0$ gives $A = -1$ and $B = 0$. Therefore we have

$$f = \eta - 1 + e^{-\eta}\cos\eta \tag{7.21}$$

and

$$h = 1 - e^{-\eta}(\cos\eta + \sin\eta). \tag{7.22}$$

Notice that $h \to 1$ as $\eta \to \infty$ as it should and $D^2 f = 0$ at $\eta = 0$, which is also a boundary condition at a free surface. In the thermal-wind main stream

$$W \sim O[a/\sigma(\ln \sigma^{-1})^{\frac{1}{2}}]$$

and in the Ekman layer $W \sim O(a)$. From (7.11) the terms in $\ln\sigma^{-1}$ give for g

$$D^6 g + 4D^2 g = 2f/(1+f^2), \tag{7.23}$$

with $g = Dg = D^6 g + 4D^2 g = 0$ at $\eta = 0$ and $g \to \frac{1}{2}\eta \ln \eta$ as $\eta \to \infty$.

The ρ convection. The solution of the isothermal equation (7.7) at $z = 0$ satisfying $W = DW = 0$ at $z = 0$ is

$$W \sim (\tfrac{1}{6}NRa^4)^{\frac{1}{2}} z^3 (\ln z^{-1})^{\frac{1}{2}}. \tag{7.24}$$

Letting $z = \epsilon\eta$ in (7.24), where ϵ is the boundary-layer thickness, gives

$$W \sim \left(\frac{NRa^4}{6}\ln\epsilon^{-1}\right)^{\frac{1}{2}} \epsilon^3 \left(\eta^3 - \frac{\eta^3 \ln\eta}{2\ln\epsilon^{-1}} + \ldots\right). \tag{7.25}$$

Therefore in the ρ thermal layer let

$$W \sim \left(\frac{NRa^4}{6}\ln\epsilon^{-1}\right)^{\frac{1}{2}} \epsilon^3 \left(f - \frac{g}{\ln\epsilon^{-1}} + \ldots\right). \tag{7.26}$$

Substituting (7.26) into (7.6) and assuming that $\epsilon^2 a^2$ and $\epsilon^4 T$ tend to zero as $\rho \to \infty$ gives

$$D^6 f + \epsilon^6 NRa^2 f = -\frac{\epsilon^6 NRa^2}{6}\ln\epsilon^{-1} f^2 D^6 f + O\left(\frac{1}{\ln\epsilon^{-1}}\right). \tag{7.27}$$

Taking $\epsilon^6 NRa^2 \ln\epsilon^{-1} = 1$ gives for f

$$D^6 f = -f^2 D^6 f, \tag{7.28}$$

with $f = Df = D^4 f = 0$ at $\eta = 0$ and $f \to n^3$ as $\eta \to \infty$.

From (3.23) and (7.2) at $\eta = 0$, $D^4 f = (\epsilon^2 T/a^2)\, Dh \to 0$ as $\rho \to \infty$. The solution to (7.28) is

$$f = \eta^3. \tag{7.29}$$

The terms in $\ln\epsilon^{-1}$ give for g

$$D^6 g = 6f/(6+f^2), \tag{7.30}$$

with $g = Dg = D^4 g = 0$ at $\eta = 0$ and $g \to \frac{1}{3}\eta^3 \ln\eta$ as $\eta \to \infty$.

Integrating (3.29) through the ρ thermal layer, the Ekman layer and the thermal-wind boundary layer gives

$$\frac{k'}{\epsilon^5 \ln\epsilon^{-1}} \sim Ra^2 - a^6 - \frac{kT^{\frac{5}{4}}}{\ln T} - \frac{2^{\frac{3}{2}} T^{\frac{5}{4}}}{\ln T}\int_0^\infty \frac{d\eta}{1+\eta^2}, \tag{7.31}$$

where

$$k' = 12\int_0^\infty \frac{d\eta}{6+\eta^6} = 2{\cdot}82326$$

by contour integration and

$$k = 2^{\frac{3}{2}} \int_{\eta=0}^{\infty} \frac{d\eta}{1+f^2} = 7{\cdot}26536,$$

where $f = \eta - 1 + e^{-\eta}\cos\eta$. From (7.31) and $\epsilon^6 N Ra^2 \ln \epsilon^{-1} = 1$,

$$\epsilon \sim 1{\cdot}698 \left[\left(Ra^2 - a^6 - 11.7082\frac{T^{\frac{5}{4}}}{\ln T}\right) \ln\left(Ra^2 - a^6 - 11{\cdot}7082\frac{T^{\frac{5}{4}}}{\ln T}\right)\right]^{-\frac{1}{5}} \tag{7.32}$$

and

$$N \sim 0{\cdot}2085 \left(1 - \frac{a^4}{R} - \frac{11{\cdot}7082}{Ra^2}\frac{T^{\frac{5}{4}}}{\ln T}\right)^{\frac{6}{5}} \left[Ra^2 \ln\left(Ra^2 - a^6 - 11{\cdot}7082\frac{T^{\frac{5}{4}}}{\ln T}\right)\right]^{\frac{1}{5}}. \tag{7.33}$$

From (7.32), for ϵ to be positive,

$$R \geqslant O(a^4) \quad Ra^2 \geqslant O(T^{\frac{5}{4}}/\ln T); \tag{7.34}$$

that is,

$$R \geqslant O[T^{\frac{5}{6}}/(\ln T)^{\frac{2}{3}}], \quad a \leqslant O[T^{\frac{5}{24}}/(\ln T)^{\frac{1}{6}}]. \tag{7.35}$$

Thus $\sigma^2 a^2 \leqslant O[1/T^{\frac{1}{12}}(\ln T)^{\frac{1}{3}}]$ and $\sigma^2 a^6/T \leqslant O(1/T^{\frac{1}{4}}\ln T)$ and tend to zero as $T \to \infty$. In the ρ isothermal main stream $W \sim O[a/\epsilon^3(\ln\epsilon^{-1})^{\frac{1}{2}}]$ and in the ρ thermal layer $W \sim O(a)$. When

$$Ra^2 \sim a^6 + 11{\cdot}7082(T^{\frac{5}{4}}/\ln T), \tag{7.36}$$

or

$$\rho A^2 \sim A^6 + 11{\cdot}7082, \tag{7.37}$$

the ρ thermal layer is thicker than the Ekman layer. However, the ρ convection only tends to zero (for a fixed T) when $A \to 0$. The T convention also tends to zero when $A \to 0$. Comparing (7.37) with (4.3) shows that the value of $\rho\,(= 11{\cdot}7082/A^2)$ is greater than $\rho_m\,(= \pi^2/A^2)$. Thus subcritical instability is not indicated for small A. From (7.33), the Nusselt number decreases as the Taylor number is increased for a given Rayleigh number.

If in (7.16) $(2\sigma^2 N Ra^2/T)\ln\sigma^{-1} = 1$ and $\sigma^2 N Ra^2 = 1$ then $\sigma^6 N Ra^2 = \sigma^4$ and $\sigma^4 T = 2\sigma^6 \ln \sigma^{-1}$. Thus neglecting these terms compared with terms of order $\sigma^2 a^2$ gives

$$D^6 f = 0, \quad D^6 g = 0, \quad \text{etc.},$$

and there is no contribution to the integral (3.29).

7.2. *Free–free boundaries*

The T convection. A solution of the thermal-wind equation (7.8) at $z = 0$ satisfying the boundary conditions $W = D^2 W = D^4 W = 0$ at $z = 0$ is

$$W \sim z - \frac{N Ra^4}{6T} z \ln\left(1 + \frac{z^2}{a^2}\right). \tag{7.38}$$

Therefore Ekman-layer solutions of the full equations (7.5) are not required. In the boundary layer of the thermal-wind equation we have as in (7.10)

$$W \sim \left(\frac{2 N Ra^4}{T}\ln\tau^{-1}\right)^{\frac{1}{2}} \tau\left(\eta - \frac{g}{\ln\tau^{-1}} + \ldots\right), \tag{7.39}$$

where τ is the boundary-layer thickness. Notice that η and g (which is given by equation (7.15)) satisfy free-surface boundary conditions. Anticipating the

ρ-thermal-layer results let $\tau^2 RNa^2 = 1$ and, as for rigid–rigid boundaries, $(2\tau^2 NRa^2/T)\ln\tau^{-1} = 1$. Thus we obtain

$$\tau = e^{-\frac{1}{2}T}. \tag{7.40}$$

The contribution from the thermal-wind boundary layer to the integral (3.29) is

$$-\frac{2T}{\tau\ln\tau^{-1}}\int_0^\infty \frac{D^2 g}{\eta}\,d\eta = -2e^{-\frac{1}{2}T}\int_0^\infty \frac{d\eta}{1+\eta^2} = -\pi e^{-\frac{1}{2}T}, \tag{7.41}$$

where (7.14) has been used.

The ρ convection. The solution at $z = 0$ to the ρ main-stream equation (7.7) satisfying the boundary conditions $W = D^2W = D^4W$ at $z = 0$ is

$$W \sim (NRa^4)^{\frac{1}{2}}\left(cz + \frac{z^5}{120c}\ln z^{-1} + \dots\right), \tag{7.42}$$

where c is a constant. Again c is independent of the wavenumber a since the terms a^2D^4W, a^4D^2W and a^6W are zero at $z = 0$. It is also independent of NRa^4, and may be determined from a numerical solution of the equation

$$(D^2-1)^3 w = -1/w, \tag{7.43}$$

with $w = D^2w = D^4w = 0$ at $z = 0, 1$.

On putting $z = \epsilon\eta$, W becomes

$$W \sim (NRa^4)^{\frac{1}{2}}\epsilon\left(c\eta + \frac{\eta^5}{120c}\epsilon^4\ln\epsilon^{-1} - \frac{\eta^5\ln\eta}{120c}\epsilon^4 + \dots\right). \tag{7.44}$$

Thus in the full equations (7.6) let

$$W \sim (NRa^4)^{\frac{1}{2}}\epsilon(f + g\epsilon^4\ln\epsilon^{-1} - p\epsilon^4 + \dots), \tag{7.45}$$

with $f \to \eta$, $g \to \eta^5/120c$ and $p \to \eta^5\ln\eta/120c$ as $\eta \to \infty$. Thus from (7.6),

$$D^6 f + \epsilon^6 NRa^2 f = -\epsilon^2 NRa^2 f^2 D^6 f + O[\epsilon^4\ln\epsilon^{-1}]. \tag{7.46}$$

If $\epsilon^2 NRa^2 = 1$ then $\epsilon^6 NRa^2 = \epsilon^4$ and the equation for f is

$$(1+f^2)\,D^6 f = 0, \tag{7.47}$$

with $f = D^2 f = D^4 f = 0$ at $\eta = 0$ and $f \to \eta$ as $\eta \to \infty$.

Therefore

$$f = c\eta. \tag{7.48}$$

The terms in $\epsilon^4\ln\epsilon^{-1}$ give for g

$$(1+f^2)\,D^6 g = 0, \tag{7.49}$$

with $g = D^2 g = D^4 g = 0$ at $\eta = 0$ and $g \to \eta^5/120c$ as $\eta \to \infty$. Thus g is given by

$$g = \eta^5/120c. \tag{7.50}$$

The terms in ϵ^4 give for p

$$D^6 p = f/(1+f^2), \tag{7.51}$$

with $p = D^2 p = D^4 p = 0$ at $\eta = 0$ and $p \to (\eta^3/120c)\ln\eta$ as $\eta \to \infty$.

The terms $\epsilon^2 a^2 D^4 f$ and $\epsilon^4 a^4 D^2 f$ in the first term on the left-hand side of (7.5) or (7.6) do not contribute to the boundary layer. Integrating (3.29) through the ρ thermal layer and the thermal-wind boundary layer gives

$$\pi/\epsilon c \sim Ra^2 - a^6 - \pi\, e^{\frac{1}{2}T}. \tag{7.52}$$

From (7.52) and the relation $\epsilon^2 N Ra^2 = 1$ it follows that

$$\epsilon \sim \pi/c)\,[Ra^2 - a^6 - \pi\, e^{\frac{1}{2}T}]^{-1} \tag{7.53}$$

and

$$N \sim \frac{c^2}{\pi^2}\frac{(Ra^2 - a^6 - \pi\, e^{\frac{1}{2}T})^2}{Ra^2}. \tag{7.54}$$

From (7.53), for ϵ to be positive, it follows that

$$R \geqslant O(a^4), \quad Ra^2 \geqslant O(e^{\frac{1}{2}T}). \tag{7.55}$$

Therefore

$$R \geqslant O(e^{\frac{1}{3}T}), \quad a \leqslant O(e^{\frac{1}{12}T}). \tag{7.56}$$

Also $\tau^2 a^2 \leqslant O(e^{-\frac{5}{6}T})$ and $a^6\tau^2/T \ll O(e^{-\frac{1}{2}T}/T)$, which justifies the assumptions made in equation (7.9). In the thermal-wind main stream $W \sim O[a/\tau(\ln \tau^{-1})^{\frac{1}{2}}]$ and in the boundary layer $W \sim O(a)$. In the ρ main stream $W \sim O[a/\epsilon]$ and in the thermal layer $W \sim O(a)$. The ρ thermal layer is thicker than the thermal-wind boundary layer for

$$Ra^2 \sim a^6 + \pi\, e^{\frac{1}{2}T}, \tag{7.57}$$

or

$$\rho A^2 \sim A^6 + \pi. \tag{7.58}$$

All convection ceases for $A \to 0$ (T fixed) and, comparing with (4.6), this occurs at lower values of ρ ($= \pi/A^2$) than ρ_m ($= \pi^2/A^2$). This indicates the possibility of subcritical instability for small A. The Nusselt number decreases as the Taylor number increases for a given Rayleigh number.

8. The thermal-wind balance

Equations (2.6) and (2.8) for steady motion, neglecting the nonlinear terms for large Prandtl number, become

$$\partial\overline{w}/\partial x = \nabla^2 u + T^{\frac{1}{2}} v, \tag{8.1}$$

$$\partial\overline{w}/\partial y = \nabla^2 v - T^{\frac{1}{2}} u, \tag{8.2}$$

$$\partial\overline{w}/\partial z = \nabla^2 w + R\theta \tag{8.3}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{8.4}$$

From (8.1) and (8.3)

$$\nabla^2 \eta + T^{\frac{1}{2}}(\partial v/\partial z) = R\,(\partial\theta/\partial x), \tag{8.5}$$

where $\eta = (\partial u/\partial z) - (\partial w/\partial x)$. As $T \to \infty$, equation (8.5) reduces to

$$T^{\frac{1}{2}}\,(\partial v/\partial z) = R\,(\partial\theta/\partial x). \tag{8.6}$$

From (8.1) and (8.2) we obtain

$$\nabla^2 \zeta = -T^{-\frac{1}{2}}\,\partial w/\partial z, \tag{8.7}$$

where $\zeta = (\partial v/\partial x) - (\partial u/\partial y)$. Substituting

$$\theta = \theta_0 + Ff, \quad v = a^{-2}[DW(\partial f/\partial y) - Z(\partial f/\partial x)], \quad w = Wf \quad \text{and} \quad \zeta = Zf$$

into (8.6) gives

$$\frac{T^{\frac{1}{2}}}{a^2}\left(D^2W\frac{\partial f}{\partial y} - DZ\frac{\partial f}{\partial x}\right) = RF\frac{\partial f}{\partial x}. \tag{8.8}$$

For two-dimensional motion in the x, z plane (8.8) reduces to

$$-(T^{\frac{1}{2}}/a^2)\,DZ = RF. \tag{8.9}$$

Since $(\partial^2 f/\partial x^2) + (\partial^2 f/\partial y^2) = -a^2 f$, equation (8.7) gives

$$(D^2 - a^2)\,Z = -T^{\frac{1}{2}}DW. \tag{8.10}$$

Thus from (8.9) and (8.10)

$$TD^2W = Ra^2(D^2 - a^2)\,F, \tag{8.11}$$

which is of the same form as equation (5.14) after replacing F by N/W.

9. Small finite amplitude solutions close to the marginal case as $T \to \infty$ $(W \ll a)$

This case can be compared with the results of Veronis (1959), who investigated solutions close to the marginal case for free–free horizontal boundaries. For $W \ll a$ equation (7.3) gives

$$F \sim NW/a^2. \tag{9.1}$$

Also, from equations (7.5) [or (7.6)],

$$(D^2 - a^2)^3\,W + TD^2W + NRa^2W \sim NRW^3, \tag{9.2}$$

or

$$(D^2 - a^2)^3\,W + TD^2W + R_0a^2W \sim (R_0a^2 - NRa^2)\,W + NRW^3, \tag{9.3}$$

where R_0 is the Rayleigh number for the marginal case and is given by

$$R_0a^2 = (\pi^2 + a^2)^3 + \pi^2T. \tag{9.4}$$

Now let $W = W_0 + W_1$, where $W_1 \ll W_0$. Substituting into (9.3) gives

$$(D^2 - a^2)^3\,W_1 + TD^2W_1 + R_0a^2W_1 = R_0a^2W_0 - NRa^2W_0 + NRW_0^3, \tag{9.5}$$

where W_0 satisfies the marginal equation (4.1).

9.1. *The solution for free–free boundaries*

In this case W_0 is given by

$$W_0 = A\cos\pi z, \tag{9.6}$$

where A is the amplitude and the boundaries are $z = \pm\frac{1}{2}$. Therefore

$$W_0^3 = A^3\cos^3\pi z = A^3(\tfrac{3}{4}\cos\pi z + \tfrac{1}{4}\cos 3\pi z). \tag{9.7}$$

Substituting (9.7) into (9.5) gives

$$\begin{aligned}(D^2 - a^2)^3\,W_1 + TD^2W_1 + R_0a^2W_1 \\ = (\tfrac{3}{4}A^2NRa^2 + R_0a^2 - NRa^2)\,A\cos\pi z + \tfrac{1}{4}NRA^3\cos 3\pi z.\end{aligned} \tag{9.8}$$

For a periodic solution the coefficient of $\cos \pi z$ must be zero and thus

$$A = \frac{2}{\sqrt{3}}\left(1-\frac{R_0}{NR}\right)^{\frac{1}{2}} a. \tag{9.9}$$

This agrees with the results of Veronis (1959) and confirms that, for R close to R_0, $W \ll a$. Letting $W_1 = B\cos 3\pi z$ in (9.8) gives

$$B = -\frac{NRa^3}{12\, 3^{\frac{1}{2}}\pi^2 T}\left(1-\frac{R_0}{NR}\right)^{\frac{3}{2}}. \tag{9.10}$$

From (9.1)

$$FW \sim \frac{NW^2}{a^2} = \frac{4N}{3}\left(1-\frac{R_0}{NR}\right)\cos^2 \pi z. \tag{9.11}$$

Substituting into (3.20) results in

$$N-1 = \int_0^1 FW\,dz = \tfrac{2}{3}(N-R_0/R). \tag{9.12}$$

If $N = 1+N_1$ then

$$N_1 = 2[1-(R_0/R)]. \tag{9.13}$$

Therefore from (9.12) it follows that

$$\int_0^1 FW\,dz = N_1, \quad \text{i.e.} \quad \frac{R}{R-R_0}\int_{z=0}^1 FW\,dz = 2. \tag{9.14}$$

Equation (9.14) also agrees with the results of Veronis (1959). For a given R as T increases R_0 will increase (from the value given by (9.4)) until it equals R. Then from (9.13) N becomes equal to 1 and (9.9) and (9.10) give A and B tending to zero.

10. Comparison

Veronis (1968) numerically solved the partial differential equations for free–free boundaries using truncated Fourier representations in the x and z directions for the vertical velocities, etc. No horizontal averages were taken. He found there was a difference in behaviour of liquids with small and large Prandtl number. For large Prandtl numbers there was no subcritical instability and the Nusselt number increased for given R/R_c (where R_c is the critical Rayleigh number) as the Taylor number increased. For small Prandtl numbers subcritical instability was predicted for Taylor numbers less than or equal to $10^{3\cdot 6}$. These effects he considered to be caused by a balance between the rotational constraint and the nonlinear inertial terms ($C \neq 0$). For larger Taylor numbers the liquid first became unstable to infinitesimal oscillatory disturbances, but a steady finite amplitude convection was established at larger values of R, which, however, are less than the values of R derived from linear theory. He found that the Nusselt number decreased as the Taylor number increased for given R/R_c. In the present paper a complete Fourier series is taken in the z direction, and although only two terms are taken in the x direction the asymptotic results are in agreement with the numerical results of Veronis.

Rossby (1969) carried out experiments with rotating fluids (water and mercury) between rigid horizontal boundaries. For water he found subcritical instability for large Taylor numbers, $T > 5 \times 10^4$. The Nusselt number reached a maximum as the Taylor number increased for a given Rayleigh number. For mercury, unsteady subcritical instability existed for finite values of the Taylor number ($0 < T < 1{\cdot}8 \times 10^4$), which agrees with the results of Veronis. Finite amplitude overstability existed for a limited range of Taylor numbers: $1.8 \times 10^4 \leqslant T \leqslant 10^5$. For larger values of T ($T > 10^5$) he found no subcritical instability but good agreement with linear overstability theory. The Nusselt number decreased monotonically as the Taylor number increased for a given Rayleigh number. The present asymptotic theory is again in accordance with Rossby's results for steady convection.

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